

Thursday, 10 April

So our starting gauge symmetry group is $SU(2) \otimes U(1)_Y$

gauge fields $SU(2)_L \otimes U(1)_Y$ & $B_{\mu\nu}$ are obtained by combination
of the above fields.

So we begin w/ 4 gauge bosons (massless)

$$SU(2)_L : W_{\mu}^1, W_{\mu}^2, W_{\mu}^3$$

$$U(1)_Y : B_{\mu\nu}$$

The total degrees of freedom are 10. Since there are 3 ∂_{μ} 's and 2 ∂_{ν} 's, there are 3 $\times 2 = 6$ degrees of freedom.

$W_{\mu\nu}^a + W_{\nu\mu}^a = 0$ and $B_{\mu\nu} + B_{\nu\mu} = 0$

$$\mathcal{L} = -\frac{1}{4} W_{\mu\nu}^a W^{a,\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

$$W_{\mu\nu}^a = \partial_{\mu} W_{\nu}^a - \partial_{\nu} W_{\mu}^a - g \epsilon^{abc} W_{\mu}^b W_{\nu}^c$$

$$B_{\mu\nu} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu}$$

The coupling to matter fields is then

$$\mathcal{L}_f = \bar{e}_R [i \gamma^{\mu} (\partial_{\mu} + ig' \frac{Y}{2} B_{\mu})] e_R$$

$$+ \bar{e}_L [i \gamma^{\mu} (\partial_{\mu} + ig' \frac{Y}{2} B_{\mu} + ig \frac{\vec{\sigma}}{2} \cdot \vec{W}_{\mu})] e_L$$

where $\vec{\sigma} = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}$ and $\vec{W}_{\mu} = \begin{pmatrix} 0 & W_{\mu}^1 \\ W_{\mu}^1 & 0 \end{pmatrix}$

~~Dirac and Majorana~~ Fermions and their combinations

These fermions are X symmetric and do not have a mass term

We have added

$2 \times 2 + 1 \times 2$ fermion dof.

$$\begin{pmatrix} v \\ e \end{pmatrix}, \begin{pmatrix} 1 \\ \downarrow \end{pmatrix}, e_R \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix}$$

Because of our chiral $SU(2) \otimes U(1)$ symmetry,
we are forbidden from adding a fermion mass term

~~Because~~ $\bar{e}_R e_L$ is not invariant, as the L & R handed fields transform independently under different gauge groups.

Now add scalar Higgs field with negative μ^2 term
to spontaneously break this symmetry

The simplest model is to add a complex doublet

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}$$

$$\phi^* = (\phi^\dagger)^T = \begin{pmatrix} \phi^- \\ \phi^0 \end{pmatrix}$$

So this Higgs field has an additional 4 dof.

$$Y_\phi = +1$$

$$Q\phi = \frac{1}{2}(\gamma_3 + \gamma) \phi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \phi : \begin{array}{l} 2 \text{ charged} \\ 2 \text{ neutral} \end{array} \text{ bosons}$$

$$\mathcal{L}_\phi = (D_\mu \phi)^+ (D^\mu \phi) - V(\phi^+ \phi)$$

$$D_\mu = \partial_\mu + ig' \frac{Y}{2} B_\mu + ig \frac{\vec{\Sigma}}{2} \cdot \vec{W}_\mu$$

$$V(\phi^+\phi) = -\mu^2 \phi^+ \phi + \lambda (\phi^+ \phi)^2$$

The symmetries still allow for an additional term which couples L & R fields

$$\mathcal{L}_{\text{eff}} = -G_F \left[\bar{e}_R \phi^+ e_L + \bar{e}_L \phi e_R \right]$$

In total, we have $8 + 6 + 4 = 18$ dof.

We then postulate the neutral component gets a vev

$$\langle \phi \rangle_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}, \quad v = \sqrt{\frac{\mu^2}{\lambda}}$$

The remaining, unbroken symmetry should correspond to a generator that annihilates the vacuum

$$G_1 \langle \phi \rangle_0 = 0$$

$$e^{i \times G_1} \langle \phi \rangle_0 = \langle \phi \rangle_0$$

In order to determine the form of G_1 , note

$$\sigma_1 \langle \phi \rangle_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = \begin{pmatrix} v\sqrt{2} \\ 0 \end{pmatrix}$$

$$\sigma_2 \langle \phi \rangle_0 = \begin{pmatrix} 0 \\ -i v/\sqrt{2} \end{pmatrix}$$

$$\sigma_3 \langle \phi \rangle_0 = \begin{pmatrix} 0 \\ -v/\sqrt{2} \end{pmatrix} = -\langle \phi \rangle_0$$

$$\gamma \langle \phi \rangle_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = \langle \phi \rangle_0$$

Recall, we assumed it was the charge neutral state with

$$Q \langle \phi \rangle_0 = \left(\frac{q}{2} + \frac{\sigma_3}{2} \right) \langle \phi \rangle_0 = 0$$

And so by construction, the generator which annihilates the vacuum and defines the remaining conserved charge is the electric charge operator. The gauge boson associated with this generator is expected to remain massless.

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Expand about vacuum

$$\phi = e^{i \vec{\xi} \cdot \frac{\vec{S}}{2v}} \begin{pmatrix} 0 \\ v+h \end{pmatrix}$$

Note: the 3rd generator should be $\kappa_3 = \bar{\kappa}_3 - Y$, the generator orthogonal to $\Omega = \frac{1}{2}(\bar{\kappa}_3 + Y)$. But since Ω annihilates the vacuum, the above is operationally identical.

In order to see the particle content of our theory, we transform to unitary gauge

$$\phi \rightarrow \phi' = e^{-i \vec{\xi} \cdot \frac{\vec{S}}{2v}} \phi = \begin{pmatrix} 0 \\ v+h \end{pmatrix}$$

$$\vec{W}_\mu \rightarrow \vec{W}'_\mu = \vec{W}_\mu + \frac{1}{gv} \partial_\mu \vec{\xi}$$

$$e_L \rightarrow e'_L = e^{i \vec{\xi} \cdot \frac{\vec{S}}{2v}} e_L$$

$$e_R \rightarrow e'_R = e_R$$

$$\vec{B}_\mu \rightarrow \vec{B}'_\mu = \vec{B}_\mu$$

In this new basis, the Yukawa terms

$$\begin{aligned} \mathcal{L}_{\phi\psi} &= -\frac{G_e}{\sqrt{2}} \frac{v+h}{\sqrt{2}} [\bar{e}_R e_L + \bar{e}_L e_R] \\ &= -\frac{G_e v}{\sqrt{2}} \bar{e} e - \frac{G_e}{\sqrt{2}} h \bar{e} e \\ &= -m_e \bar{e} e - \frac{m_e}{v} h \bar{e} e \end{aligned}$$

So we see the electron mass comes from the Yukawa coupling to the Higgs, and with the VEV (vacuum expectation value)

$$m_e = G_e$$

$$m_e = \frac{y_e v}{\sqrt{2}}$$

$$\begin{aligned} v &\approx 246 \text{ GeV} \\ &= 246 \times 10^3 \text{ MeV} \end{aligned}$$

$$\begin{aligned} m_e &= 0.511 \text{ MeV} \\ y_e &= 2.94 \times 10^{-6} \end{aligned}$$

Compare with the top quark mass/coupling

$$m_t = \frac{y_t v}{\sqrt{2}}$$

$$m_t \approx 173 \text{ GeV}$$

$$y_t \approx 0.995$$

There is a vast hierarchy of these couplings.

If the neutrinos get their masses via the Higgs also, the hierarchy is even more orders of magnitude.

The values of these couplings are just parameters of the Standard Model, and we currently have no verified ideas how to predict them.

What are the quadratic terms in the \mathcal{L}_0 ?

$$\mathcal{L}_{\text{quad}} = \frac{1}{2} \left[\partial_\mu h \partial^\mu h - 2|u^2| h^2 \right] + \frac{v^2}{8} \left[g^2 |W_\mu^1 - iW_\mu^2|^2 + (g' B_\mu - g W_\mu^3)^2 \right]$$

We can re-write the field operators in terms of more familiar charge eigenstates

$$W_\mu^\pm = \frac{W_\mu^1 \mp iW_\mu^2}{\sqrt{2}} = (W_\mu^\pm)^\dagger$$

$$t^\pm = \frac{1}{2} (\sigma^1 \pm i\sigma^2), \quad t^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$W_\mu = \sum_a t^a W_\mu^a = \begin{pmatrix} \frac{W^0}{2} & W^+ \\ W^- & -\frac{W^0}{2} \end{pmatrix}$$

$$\mathcal{L}_{\text{quad}}^{W^\pm} = \frac{g^2 v^2}{4} W_\mu^{+\dagger} W^{-\mu\dagger} = \frac{g^2 v^2}{4} W_\mu^- W^{\mu\dagger}$$

$$= \frac{g^2 v^2}{4} \frac{W_\mu^- W^{-\mu\dagger} + W_\mu^+ W^{\mu\dagger}}{2}$$

And so our vector bosons have a mass given by the Higgs vev

$$M_W = \frac{gv}{2}$$

What about the neutral vector boson?

We see it is a mixture of the 3rd component of the SU(2) gauge boson and the U(1)_Y boson.

We can express the mixing in terms of a mixing angle

$$\tan \theta_W = \frac{g'}{g}$$

$$Z_\mu = \frac{-g' B_\mu + g W_\mu^3}{\sqrt{g'^2 + g^2}} = -\sin \theta_W B_\mu + \cos \theta_W W_\mu^3$$

We then see the mass is

$$\begin{aligned} M_Z^2 &= \frac{v^2}{4} (g'^2 + g^2) = \frac{g^2 v^2}{4} \left(1 + \frac{g'^2}{g^2} \right) \\ &= \frac{g^2 v^2}{4} \left(1 + \tan^2 \theta_W \right) \\ &= \frac{g^2 v^2}{4 \cos^2 \theta_W} \\ &= \frac{M_W^2}{\cos^2 \theta_W} \end{aligned}$$

We have the orthogonal neutral vector boson

$$A_\mu = \frac{g B_\mu + g' W_\mu^3}{\sqrt{g'^2 + g^2}} = \cos \theta_W B_\mu + \sin \theta_W W_\mu^3$$

Which does not acquire a mass

So the quadratic \mathcal{L} + Yukawa Interactions, after SSB, in Unitary - Gauge

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4} W_{\mu\nu}^a W^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\ & + (\partial_\mu \phi)(\partial^\mu \phi) - V(\phi + \phi) \\ & + \bar{e}_R [i\gamma^\mu (\partial_\mu + ig' \frac{Y}{2} B_\mu)] e_R \\ & + \bar{e}_L [i\gamma^\mu (\partial_\mu + ig' \frac{Y}{2} B_\mu + ig \frac{\vec{\sigma}}{2} \cdot \vec{W}_\mu)] e_L \\ & - \gamma_e [\bar{e}_R \phi^+ e_L + \bar{e}_L \phi e_R]\end{aligned}$$



$$\begin{aligned}& -\frac{1}{4} \tilde{W}_{\mu\nu}^a \tilde{W}^{a\mu\nu} + \frac{1}{2} [M_W^2 |W_\mu^+|^2 + M_W^2 |W_\mu^-|^2 + M_Z^2 |\tilde{Z}_\mu|^2] \\ & - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ & + \frac{1}{2} [\partial_\mu h \partial^\mu h - 2 \mu^2 |h|^2] \\ & - \frac{g}{\sqrt{2}} \bar{e}_L \gamma^\mu (\sigma^+ W_\mu^+ + \sigma^- W_\mu^-) e_L + \mathcal{L}_{NC} \\ & + \bar{e} (\gamma^\mu i(\partial_\mu + \frac{gg'}{\sqrt{g'^2+g^2}} (-i) A_\mu)) e - \bar{e} m_e e \quad \text{↑ neutral current} \\ & + \dots\end{aligned}$$

$$\mathcal{L}_{NC} = e \bar{e} \gamma^\mu A_\mu e - \frac{g}{2 \sin \theta_W} \bar{e}_L \gamma^\mu Z_\mu v_L$$

$$\frac{-g}{2 \cos \theta_W} \left[2 \sin^2 \theta_W \bar{e}_R \gamma^\mu e_R + (2 \sin^2 \theta_W - 1) \bar{e}_L \gamma^\mu e_L \right] Z_\mu$$

$$g = 2 M_W \sqrt{f_2 G_F} \quad g' = \frac{\sin \theta_W}{\cos \theta_W} g$$

We see - as we expect

- the coupling to the vector field of remaining $U(1)$ is purely vector
- the coupling of Z to ν is purely left handed
- the coupling of Z to e has L & R couplings but with unequal strengths

$$-\sqrt{\frac{G_F M_Z^2}{2\sqrt{2}}} \left[(4 \sin^2 \theta_W - 1) \bar{e} \gamma^\mu e + \bar{e} \gamma^\mu \gamma^5 e \right] Z_\mu$$

$$\sin^2 \theta_W = 1 - \frac{M_W^2}{M_Z^2}$$

$$= 1 - \frac{(80 \text{ GeV})^2}{(91.2 \text{ GeV})^2}$$

$$\approx 0.23$$

So we see the neutral current is almost purely axial-Vector (A).

You may have noticed we chose "unitary gauge" to expand about the vacuum

$$\phi \rightarrow \phi' = e^{-i\vec{\xi} \cdot \frac{\vec{v}}{2v}} \phi = \begin{pmatrix} 0 \\ v+h \end{pmatrix}$$

$$\vec{W}_\mu \rightarrow \vec{W}'_\mu = \vec{W}_\mu + \frac{1}{gv} \partial_\mu \vec{\xi}$$

$$B_\mu \rightarrow B'_\mu = B_\mu$$

$$e_L \rightarrow e'_L = e_L^{-i\vec{\xi} \cdot \frac{\vec{v}}{2v}}$$

$$e_R \rightarrow e'_R = e_R$$

- This choice makes the "particle content" clear.

This choice makes calculations tricky. The massive vector fields ~~were~~ give rise to very singular behavior (renormalization requires more careful algebra).

- Also, note, we picked a gauge where there was no evidence of Goldstone modes!

This should make you realize the story of SSB w/ gauge interactions is more subtle.

Gauge Symmetry \neq continuous symmetry
 = redundancy in description of dof.

So "breaking" gauge symmetry does not lead to goldstone modes. But this language has been rooted in place.

Another way to see the issue, we chose
a new

$$\langle \phi \rangle = \phi'$$

$$= \langle (\frac{\phi}{\sqrt{2}}) \rangle$$

But this expectation is an average over the
gauge fields. $\langle v \rangle = 0$ because it is not
a gauge invariant quantity.

- A gauge symmetry forces upon us massless modes
- Vacuum does not respect gauge invariance
 ↳ massless mode becomes massive
- global symmetry is not

Coming back to the \mathcal{L}

$$\mathcal{L} \supset (D_\mu \phi)^+ (D^\mu \phi) - V(\phi^+ \phi)$$

$$D_\mu \phi = (\partial_\mu + i g \frac{\sqrt{2}}{2} \vec{W}_\mu + i g' \frac{\sqrt{2}}{2} \vec{B}_\mu) \phi$$

$$V(\phi^+ \phi) = \mu^2 \phi^+ \phi + \lambda (\phi^+ \phi)^4$$

$$\mu^2 < 0$$

$$\text{We take } \langle \phi^+ \phi \rangle = \frac{v^2}{2}$$

$$\text{and pick the direction } \langle \phi \rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}, v = \sqrt{-\frac{\mu^2}{\lambda}}$$

Which leads to expanding about the vacuum

$$\phi = \begin{pmatrix} \phi_1 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \frac{v}{\sqrt{2}} + \phi_2 \end{pmatrix} = e^{i \frac{\sqrt{2}}{2} \frac{\vec{s}}{v}} \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix}$$

$$V(\phi^+ \phi) = +\mu^2 \phi^+ \phi + \lambda (\phi^+ \phi)^4$$

$$= \mu^2 \left[\phi'^+ \phi' + \langle \phi \rangle^+ \phi' + \phi^+ \langle \phi \rangle + \langle \phi \rangle^+ \langle \phi \rangle \right] + \lambda (\phi^+ \phi)^4$$

If we focus on quadratic part of potential

$$V(\phi^+ \phi) \supset \mu^2 \phi'^+ \phi' + \lambda v^2 \phi'^+ \phi' + \frac{\lambda v^2}{2} (\phi_2'^+ + \phi_2'^+) \quad \text{recal } v^2 = -\frac{\mu^2}{\lambda}$$

$$= \frac{\mu^2}{2} (\phi_2'^+ + \phi_2'^+)^2$$

$$\frac{1}{\sqrt{2}} (\phi_1' + \phi_1'^+), \frac{i}{\sqrt{2}} (\phi_1' - \phi_1'^+), \frac{i}{\sqrt{2}} (\phi_2' - \phi_2'^+) \quad \text{"massless" modes}$$

34) But these massless modes are actually coupled to the massive Vector W_μ bosons

$$(D_\mu \phi)^T (D^\mu \phi) = \partial_\mu \phi^T i g \frac{\vec{\sigma}}{2} \vec{W}^\mu \phi - i g \frac{\vec{\sigma}}{2} W_\mu \phi^T \partial^\mu \phi \\ = \partial_\mu \phi^T i g \frac{\vec{\sigma}}{2} \vec{W}^\mu \left(\begin{array}{c} 0 \\ v/\sqrt{2} \end{array} \right) - i g \frac{\vec{\sigma}}{2} W_\mu \left(\begin{array}{c} 0 \\ v/\sqrt{2} \end{array} \right) \partial^\mu \phi$$

So we see these interactions, in fact mean the "goldstone modes" and the vector bosons are the same

Using $\phi = e^{i \frac{\vec{\sigma}}{2v} \cdot \vec{r}} \left(\begin{array}{c} 0 \\ v+h \end{array} \right)$

makes this transparent. You can then see, a simple gauge transformation

$$W_\mu \rightarrow W_\mu + \partial_\mu \vec{z} \frac{\vec{\sigma}}{2v}$$

removes these interactions (and massless modes) from the Lagrangian. \rightarrow This is the "unitary gauge" we did earlier.